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| RESEARCH ARTICLE

Order Statistics and ML Estimators of a 3-Component Mixture of Power Distribution: A Simulation Study

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| ABSTRACT

The probability density functions of k^{th}, 1^{st} and n^{th} order statistics are explored in this study. Also, the r^{th} moments, their mean and variance of k^{th}, 1^{st} and n^{th} order statistics are derived. The Maximum Likelihood (ML) method is employed to estimate the parameters, and a system of non-linear equations is solved to derive the limiting expressions of the ML estimators and their variances, utilizing Fisher's information matrix. A detailed numerical analysis of the ML estimators' performance is carried out through a Monte Carlo simulation for various sample sizes, test termination times, and parameter values. Additionally, the practical applicability of the 3-component mixture of Power distributions is demonstrated by estimating the ML parameters using real-world data. The study's results indicated that maximum likelihood estimators based on complete (uncensored) data perform significantly better than those based on censored data. Additionally, the estimators derived from uncensored data are more dependable and precise.

| KEYWORDS

Maximum likelihood estimation, Order statistics, Power Distribution, Monte Carlo Method

| ARTICLE INFORMATION

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1. Introduction

A mixture distribution arises naturally when a statistical population consists of several subpopulations. The finite combined lifetime distribution is particularly useful in business. Mixture models combine different distributions, where the probability of each component is called its mixture weight. This model is valuable when the population distribution is unknown, making the standard distribution inadequate. For example, the lifetime of electrical components or chemicals can be modeled as a mixed distribution based on different failure factors. The distribution of the mixture may be univariate or multivariate, and all components may be discrete or continuous.

A full mixing model is required when there is no intensive mixing or distribution process, but the entire mixture is mixed. Mixed models allow direct use of distribution models or indirect use in cluster analysis, primarily density estimation ecology Bhattacharya (1967), medicine Chivers (1977), and social sciences (Harris, 1983), financing (Jedidi et al., 1997), reliability (Sultan, 2007), and industrial engineering (Ali, 2012). With advances in statistical techniques, hybrid models are widely used in various fields to model complex data.

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In many practical situations, the distribution of factors at a finite mixing rate is known, but the mixing rate is unknown, or vice versa. Sometimes, component distributions and their parameters are unknown. Component densities may belong to the same classification family or to another. In this thesis, we focus on the classical analysis of 3-component mixtures of Power distributions. This includes modeling unknown parameters of component density and mixing ratios under a type I mixing model, where the components fit within the same family of parameters.

If X has the following density function, *it* can be written in the form $f(x)=\sum_{i=1}^{k} w_i f_i(x)$, where w_i $(i=1,\ 2,\ldots,\ \mathrm{K})$ is i^{th} mixing proportions such that $w_k=1-\sum_{i=1}^{k-1} w_i$ and $f_i(x)$ is i^{th} component density function then X random variable is supposed a finite mixture of distribution to follow with a k components. With unknown mixing proportions w_1 and w_2, Interpret the resulting acceleration (PDF) and the Cumulative Spread function (CDF) of a compound by Barger (2006) and by Stˇrelec and Stehlík (2012) as:

$$f(x;\Psi)=w_1 f_1(x;\beta_1)+w_2 f_2(x;\beta_2)+(1-w_1-w_2)f_3(x;\beta_3),\ 0<x<1 \qquad (1.1)$$
$$F(x;\Psi)=w_1 F_1(x;\beta_1)+w_2 F_2(x;\beta_2)+(1-w_1-w_2)F_3(x;\beta_3) \qquad (1.2)$$

Where $w_i\geq 0,\ \sum_{i=1}^{2} w_i\leq 1,\ \Psi=(\beta_1,\beta_2,\beta_3,w_1,w_2)$ and, $f_i(x;\Psi_i),\ i=1,\ 2,\ 3$ is the pdf of the i^{th} component.

In daily life, various types of data are encountered, including simple, grouped, censored, progressively censored, and record values. Censoring occurs when only partial information about an observation is known, often in lifetime studies. Censorship is not a parameter but a feature of the product. The main analysis methods are right analysis, left analysis, and periodic analysis. The review policy is further divided into Type I and Type II, where the number or time of failure is specified in advance. Truncation, unlike censoring, omits data outside a range. Researchers like Sindhu, Feroze, and Aslam have extensively studied censored and doubly censored sampling schemes in statistical distributions.

The comprehensive literature on mixture distributions is given. In material and methods, a general discussion of the definition and formulas of statistical properties, reliability properties, inequality measures, entropies, and order statistics was given. The Results and discussion is devoted to the characterization of a three-component mixture of Power distributions. Initially, common properties such as the mean the r^{th} moment, calculate the mean, mode, moment function, and eigenfunction. Different properties such as survival rate, hazard rate, recurrence rate, hazard rate, median survival rate, and median waiting time are also derived. Prepare these beliefs and values to evaluate their behavior and generalized entropy index. Information measures of different distributions such as Shannon entropy, Rainey entropy and -entropy are also obtained. k^{th}, n^{th}, and 1^{st} order statistics and their mean variance and r^{th} moments are also derived. The parametric estimation is also discussed at the end of this chapter.

1.1 Objectives

The first major goal of this study is the classical estimation of three-point mixture energy distributions. The second goal of this study is to increase the reliability of the classification of composites considered as three elements.

Some other important aims are:

- To derive the different expressions of order statistics of 3-component mixture of Power distribution.
- To estimate the component and mixing proportion parameters. $(\beta_1,\beta_2,\beta_3,w_1,w_2)$ of the 3-component mixture of Power distribution through MLE.
- To check the over-estimation and under-estimation of the component and the parameters of mixing proportion $(\beta_1,\beta_2,\beta_3,w_1,w_2)$.

- To check the impact of sample size and the test termination time *T* on the ML estimators.
- To derive the limiting expressions for ML estimators when test termination time *T* tends to be one.

2. Review of Literature

Saleem & Aslam (2009) derived Bayes estimates for the parameter confidence function of the power distribution, using squared-error and linear, exponential loss functions and comparing Bayesian estimators to squared error loss in terms of risk, supported by statistical analysis. Shawky & Bakoban (2009) estimated the failure rate function, reliability, and coefficient for a finite mixture of two exponential gamma distributions. Bayes and maximum likelihood methods were used, and the results of both statistical methods were compared using Lindley's approximation and Monte Carlo simulations.

According to Amin (2011), the ML estimation method based on type 1 censored data was used to estimate the variance of the compound Rayleigh distribution. Analyze and obtain ML estimates of unknown parameters from Fisher's data matrix. A numerical example is carried out to illustrate the theoretical results.

Dey & Maiti (2012) analyzed a Rayleigh distribution and derived its Bayes estimators under assumptions of a symmetric loss function and an asymmetric loss function; its associated risk is derived, which is based on extended Jeffrey's prior. For the parameter, they also derived the highest posterior density and equal-tail prediction intervals and, for future observations, the highest posterior density prediction intervals. To estimate the results of Bayes estimates in many situations, the Monte Carlo simulations are performed. Finally, to learn about the real data application of the Rayleigh distribution, an illustrative example is shown.

Majeed et al. (2013) studied a two-component mixture of an inverted exponential distribution by taking it as a sense of life distribution. For a mixture model, Bayes estimators and its posterior risk under quadratic loss function for unknown parameters θ_1 and θ_2 and mixing proportion p are derived. The comparative study of Bayes estimators is considered improper, uniform, and informative prior. As a function, the test termination time of the Bayes estimators, its posterior risk, and an ML estimator method are analyzed. The numerical expressions of estimates are obtained in the condition of an infinite termination time test. Finally, to illustrate the results, a numerical study is viewed, and a mixture of data examples is simulated.

Ahmed et al. (2013) analyzed power distributions, emphasizing their applications in contact theory, wind speed modeling, and physics. ML statistics and Bayes statistics were used to estimate the classification parameters under different loss functions, and the results were analyzed with mean squared errors. Merovci & Sekiraça (2013) extended the Rayleigh distribution, studying its numerical properties and actual data processing using transformation mapping. Cooking (2013) calculated Rayleigh distribution parameters under a type-II double-sensor sampling scheme, obtained asymptotic variances by ML calculations, and tested them with Monte Carlo simulations.

Prakash (2015) analyzed the comparative study, which depends on two different loss functions, which are asymmetric. For the present study, the Rayleigh model of two-parameter is mentioned here as the underlying model, which evaluates the Bayes estimator's properties in a right progressive Type-II censored data.

Aslam et al. (2015) have worked on a 3-component mixture of Exponential distribution from a Bayesian perspective. It is very popular in survival analysis and reliability theory; due to this, the censored sampling environment is considered. Under different scenarios, the expression of the Bayes estimator and its posterior risks. In this case, the available information of prior is less, and elicitation of the hyperparameters is also given. For numerical results, the informative and non-informative prior performance of Bayes estimators on many loss functions was used, and a simulated procedure was used on their statistical goods for varying sample sizes and termination times of the test. For practical purposes, an example that depends on real-life data on engineering is also presented.

Abd-Elmougod & Mahmoud (2016) studied a two-parameter compound Rayleigh distribution in hybrid censoring with a growth type-II model and partial permanence test; the probability equations were numerically solved by maximum likelihood (ML) estimation and estimated bootstrap confidence intervals offered a suggestion. Their results were evaluated using simulated data and Monte Carlo simulations to compare confidence interval performance.

Aslam et al. (2018) performed a reliability analysis for 3-CMED, 3-CMRD, 3-CMBD, and 3-CMPD, derived from CDF, HRF, MRL, and MWT functions, with numerical results for fixed parameter values.

3. Material and Methods

3.1 Power Distribution

Censoring plays an important role in lifecycle data analysis because, in real-life situations, it is impractical to observe all events until they are completed. The scanned cases contain at least one observed case with only partial failure information. Censoring can be divided into three categories: right, gap, and left. Eligible analyses include Type I (random failures) and Type II (pre-selected failure rates). Prentice and Kalbfleisch (2002) define censoring. Finite mixture models are often used to study unknown subpopulations, such as the lifetime profile of electrical components (Davis, 1952). Acheson & McElwee (1952) found that failure in electronic tubes can occur for a variety of reasons. Sinha (1998) used a Bayesian approach to study mixtures, (Meniconi & Barry, 1996) examined the reliability of electrical components through power-function classification (Shanker et al., 2017) analyzed the Lindley distribution, while (Kamaruzzaman et al., 2012) proposed a normal distribution mix for stock market indices.

The behavior of the energy distribution for different products of the measured value is shown in Figure 1.

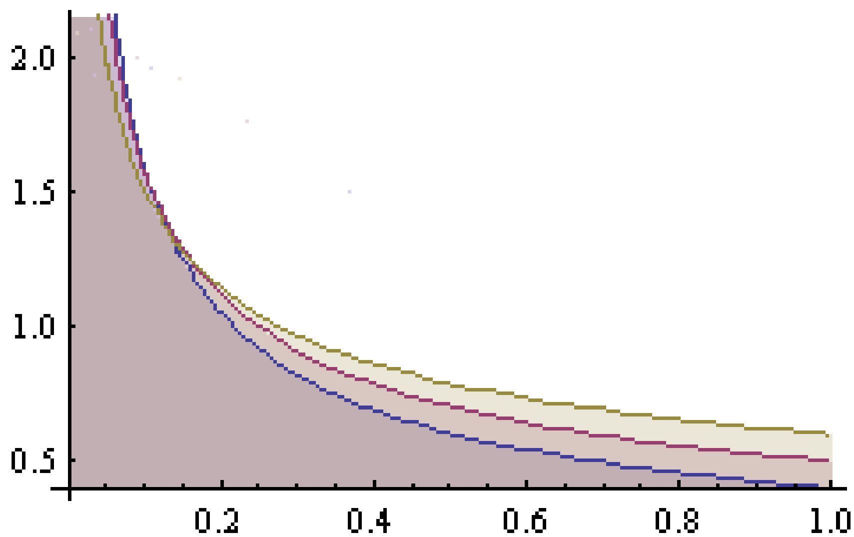

Figure 3.1: Graph of PDF of Power distribution for parameters $(\beta_1, \beta_2, \beta_3)$ = (0.5, 1, 2).

3.1.1 Uses of Power Distribution

i. In communications theory, Power distribution is used to describe the several paths of the scattered dense signals that reach a receiver.
ii. In physical sciences, Power distribution is used to observe the speed of wind, sound, or light radiations and the wave heights.
iii. In engineering, Power distribution is used to note down an object's lifetime, where the object's lifetime is based on the age of an object, e.g., the resistors, the transformers, and the capacitors in sets of an aircraft radar.
iv. In medical imaging science, Power distribution is used to demonstrate the noise variation in the imaging of magnetic resonance.

3.1.2 Order Statistics
An Order statistic deals with the applications and the properties of a r.v. which are ranked. When natural problems are studied, in which a flood, a longevity, a breaking strength, an atmospheric pressure of wind, etc., by use of order statistics which are important or applicable in a manner that, in these cases, a problem in which interest is present reduced towards the extreme observations. For example, the system is powered by six batteries, and the system is backed up when the third series dies. In that case, it can be said that the distribution of the third account will be known. In this sense, it is assumed that when a secondary battery dies, the system does not show less efficiency and hits casts financially every time, so it goes that way. In this case, we need to know about the range distribution between the second process and the third process. Thus, this example illustrates the use of sequential mathematics in different classes.

3.1.3 Probability density function of k^{th} order statistic
The k^{th} order statistic belonging to a statistical sample is identical to its k^{th}-smallest value. Pdf of k^{th} order statistics of a *X* r.v. is given as:

$$g(x_{k:n}) = \frac{n!}{(k-1)!(n-k)!}\{F(x)\}^{k\text{-}1}\{1-F(x)\}^{n-k} f(x). \qquad (3.1)$$

3.1.4 Estimation of Parameters
The estimation of parameters is a more precise method that is used for assessing the cost and an interval (duration), and it uses the association among the variables to estimate the cost or an interval. The approximation of parameters is determined by recognizing the cost of a unit or an interval and several units that are essential for the plan or activity. Here, we discussed ML estimation under censored and complete sampling situations. The ML estimator in the full sample condition can be obtained arbitrarily by assuming equal likelihood with multiple censoring times.

3.1.5 Method of Maximum Likelihood (ML)
In 1922, Sir Ronald A. Fisher developed the method of ML. The underlying logic of ML is to study each probable value that the parameter might have, and for every value, the probability is calculated that an assumed sample has happened if that were the true value of the parameter. The parametric value, which has the greatest probability of being an assumed sample, is selected as an estimate. The procedure for finding the ML estimators is given below:
Let $x_1,\ \mathrm{x}_2,\ldots,x_n$ be a r.s. from probability distribution $f\left(x;\ \beta\right)$ where β is a vector of parameter and the joint distribution for $x = x_1,\ \mathrm{x}_2,\ldots,x_n$ is as:

$$f\left(x;\ \beta\right) = f\left(x_1;\ \beta\right), f\left(x_2;\ \beta\right),\ldots, f\left(x_n;\ \beta\right), \qquad (3.2)$$

The LF is a function of the parameters of a statistical model given data. It plays a key role in statistical inference, particularly in the methods of estimating the parameter from a set of statistics. The combined distribution observed as a function of the parameter β is said to be a likelihood function of the sample. It is denoted as:

$$L\left(\mathbf{x};\ \beta\right) = \prod_{i=1}^{n} f\left(x_i;\ \beta\right), \qquad (3.3)$$

and the log LF is:

$$\partial l = \ln L\left(x;\ \beta\right) = \sum_{i=1}^{n} \ln f\left(x_i;\ \beta\right), \qquad (3.4)$$

ML estimators are found by solving the following non-linear equation:

$$\frac{\partial l}{\partial \beta} = 0. \qquad (3.5)$$

for which

$$\frac{\partial^2 l}{\partial \beta^2} < 0. \quad (3.6)$$

3.1.6 Properties of ML estimator

The properties of ML estimators are given below:

i. ML estimators are always efficient and consistent.
ii. For sufficient statistics of parameter, they are sufficient.
iii. They are generally biased.
iv. For a large sample size, they are approximately normally distributed.

3.1.7 Uses of ML estimator

ML estimators are used for various models of statistics, namely,

i. Linear and generalized linear models.
ii. Structural equation modelling.
iii. Confirmatory and exploratory factor analysis.
iv. And in many situations in the testing of hypotheses and formation of confidence intervals.

3.1.8 Advantages of the ML estimator

The advantages of ML estimators are as follows:

i. The ML estimators have a consistent approach to the difficult estimation of parameters. This means that for a huge variety of situations, ML estimates can be developed. As an example, these estimates can be used to censor data in a reliability analysis under the different models of censoring.
ii. As the sample size increases, they become a minimum variance unbiased estimator.
iii. They have an approximate variance of the sample and an approximate normal distribution, which can be applied to generate the confidence bounds and test the parameter's hypothesis.

4. Order Statistics and ML Estimators of a 3-Component Mixture of Power Distribution

4.1 Order Statistics of the 3-CMPD

In order statistics, the density of the k^{th} order statistic $x_{k:n}$ say $g(x_{K:n};\ \Psi)$, in a r.s. of size n from a 3-CMPD and the expressions for the r^{th} raw moments, mean, and the variance of the first and nth order statistics are derived as:

4.1.1 Pdf of k^{th} order statistic of the 3-CMPD

The k^{th} order statistic for a statistical sample is equal to its k^{th} smallest value. The pdf of k^{th} order statistics is denoted by $I(x_{k:n};\ \Psi)$ and evaluated as:

$$I(x_{K:n};\ \Psi) = \frac{n!}{(k-1)!(n-k)!}\{1 - R(x;\ \Psi)\}^{k-1}\{R(x;\ \Psi)\}^{n-k}\mathrm{f}(x;\ \Psi)\ , \quad (4.1)$$

where

$$f(x;\ \Psi) = w_1 f_1(x;\beta_1) + w_2 f_2(x;\beta_2) + (1 - w_1 - w_2) f_3(x;\beta_3) \ \ w_1, w_2 \geq 0,\ \ w_1 + w_2 \leq 1.$$

$$1 - R(x;\ \Psi) = w_1\big(1 - R(x;\beta_1)\big) + w_2\big(1 - R(x;\beta_2)\big) + (1 - w_1 - w_2)\big(1 - R(x;\beta_3)\big)\ ,$$

$$f_i(x;\beta_i) = \beta_i x^{\beta_i - 1},\ 0 < x < 1,\ \beta_i > 0, i = 1, 2, 3,$$

$$1 - R_i(x;\beta_i) = 1 - x^{\beta_i}, 0 < x < 1, \beta_i > 0, i = 1,2,3\ ,$$

$$\{1 - R(x;\ \Psi)\}^{k-1} = \left\{1 - w_1 x^{\beta_1} - w_2 x^{\beta_2} - (1 - w_1 - w_2) x^{\beta_3}\right\}^{k-1} \quad (4.2)$$

After simplification, equation (4.64) becomes:

$$\{1-R(x;\ \Psi)\}^{k-1}=\sum_{j=0}^{k-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)\binom{k-1}{j}\binom{j}{l}\binom{l}{r}\{\beta_1(j-l)x\}\{\beta_2(l-r)x\}\ \{\beta_3(r)x\}w_1^{j-l}w_2^{l-r}(1-w_1-w_2)^r \quad (4.3)$$

$$\{R(x;\ \Psi)\}^{n-k}=\left\{w_1x^{\beta_1}+w_2x^{\beta_2}+(1-w_1-w_2)x^{\beta_3}\right\}^{n-k}, \quad (4.4)$$

After simplification, equation (4.66) becomes:

$$\{1-R(x;\ \Psi)\}^{k-1}=\sum_{U=0}^{n-k}\sum_{v=0}^{U}\binom{n-k}{e}\binom{e}{f}\{\beta_1(n-k-e)x\}\{\beta_2(e-f)x\}.\ \ \{\beta_3(f)x\}w_1^{n-k-e}w_2^{e-f}(1-w_1-w_2)^f \quad (4.5)$$

By using the results of equations (4.3) and (4.5) in (4.1), the pdf of k^{th} order statistic for a *X* random variable becomes:

$$I(x_{k:n};\ \Psi)=\frac{n!}{(k-1)!(n-k)!}\sum_{j=0}^{k-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j\binom{k-1}{j}\binom{j}{l}\binom{l}{r}\{\beta_1(j-l)x_k\}\{\beta_2(l-r)x_k\}\{\beta_3(r)x_k\}w_1^{j-l}w_2^{l-r}\ (1-w_1-w_2)^r\ \sum_{e=0}^{n-k}\sum_{f=0}^{e}\binom{n-k}{e}\binom{e}{f}\{\beta_1(n-k-e)x_k\}\{\beta_2(e-f)x_k\}\ \ \{\beta_3(f)x\}w_1^{n-k-e}w_2^{e-f}(1-w_1-w_2)^f\left\{w_1\beta_1x_k^{\beta_1-1}+w_2\beta_2x_k^{\beta_2-1}+(1-w_1-w_2)\beta_3x_k^{\beta_3-1}\right\} \quad (4.6)$$

$$I(x_{k:n};\ \Psi)=\frac{n!}{(k-1)!(n-k)!}\sum_{j=0}^{k-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j\binom{k-1}{j}\binom{j}{l}\binom{l}{r}\ \{\psi_{01}x_k\}w_1^{\alpha_{01}-1}w_2^{\beta_{01}-1}(1-w_1-w_2)^{\gamma_{01}-1}\sum_{e=0}^{n-k}\sum_{f=0}^{e}\binom{n-k}{e}\binom{e}{f}\{\psi_{02}x_k\}w_1^{\alpha_{02}-1}w_2^{\beta_{02}-1}(1-w_1-w_2)^{\gamma_{02}-1}\left\{w_1\beta_1x_k^{\beta_1-1}+w_2\beta_2x_k^{\beta_2-1}+(1-w_1-w_2)\beta_3x_k^{\beta_3-1}\right\} \quad (4.7)$$

where

$\psi_{01}=\beta_1(j-l)+\beta_2(l-r)+\beta_3(r), \alpha_{01}=j-l+1, \beta_{01}=l-r+1, \gamma_{01}=l+1,$

$\psi_{02}=\beta_1(n-k-e)+\beta_2(e-f)+\beta_3(f), \alpha_{02}=n-k-e+1, \beta_{02}=e-f+1, \gamma_{02}=f+1.$

4.1.2 Pdf of 1^{st} order statistic of the 3-CMPD

By putting, *K* = 1 in equation (4.6), the pdf of 1^{st} order statistic becomes:

$$I(x_{1:n};\ \Psi)\ n\sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f}\{\beta_1(n-l-e)x_1\}\{\beta_2(e-f)x_1\}\{\beta_3(f)x_1\}\ w_1^{n-1-e}w_2^{e-f}(1-w_1-w_2)^f\left\{w_1\beta_1x_1^{\beta_1-1}+w_2\beta_2x_1^{\beta_2-1}+(1-w_1-w_2)\beta_3x_1^{\beta_3-1}\right\}, \quad (4.8)$$

$$I(x_{1:n};\ \Psi)\ n\sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f}\left\{\left(\beta_1(n-l-e)+\beta_2(e-f)+\beta_3(f)\right)x_1\right\}\ w_1^{n-1-e}w_2^{e-f}(1-w_1-w_2)^f\left\{w_1\beta_1x_1^{\beta_1-1}+w_2\beta_2x_1^{\beta_2-1}+(1-w_1-w_2)\beta_3x_1^{\beta_3-1}\right\},$$

$$I(x_{1:n};\ \Psi)=n\Big[\beta_1\sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f}\left\{\left(\beta_1(n-1-e)+\beta_2(e-f)+\beta_3(f)\right)x_1\right\}\ w_1^{n-1-e+1}w_2^{e-f+1-1}(1-w_1-w_2)^{f+1-1}+\beta_2\sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f}\left\{\left(\beta_1(n-1-e)+\beta_2(e-f+1)+\beta_3(f)\right)x_1\right\}\ w_1^{n-1-e+1}w_2^{e-f+2-1}(1-w_1-w_2)^{f+1-1}+\beta_3\sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f}\left\{\left(\beta_1(n-1-e)+\beta_2(e-f)+\beta_3(f+1)\right)x_1\right\}\ w_1^{n-1-e+1}w_2^{e-f+1-1}(1-w_1-w_2)^{f+2-1}\Big],$$

$$I(x_{1:n};\ \Psi)=nx_1\Big[\beta_1\sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f}\{\psi_{11}x_1\}w_1^{\alpha_{11}-1}w_2^{\beta_{11}-1}(1-w_1-w_2)^{\gamma_{11}-1}+\beta_2\sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f}\{\psi_{12}x_1\}w_1^{\alpha_{12}-1}w_2^{\beta_{12}-1}(1-w_1-w_2)^{\gamma_{12}-1}+\beta_3\sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f}\{\psi_{13}x_1\}w_1^{\alpha_{13}-1}w_2^{\beta_{13}-1}(1-w_1-w_2)^{\gamma_{13}-1}\Big]. \quad (4.9)$$

where

$\psi_{11}=\beta_1(n-e)+\beta_2(e-f)+\beta_3(f),\ \alpha_{11}=n+1-e, \beta_{11}=e-f+1, \gamma_{11}=f+1,$

$\psi_{12} = \beta_1(n-1-e) + \beta_2(e-f+1) + \beta_3(f),\ \alpha_{12} = n-e, \beta_{12} = e-f+2, \gamma_{12} = f+1,$

$\psi_{13} = \beta_1(n-1-e) + \beta_2(e-f) + \beta_3(f+1),\ \alpha_{13} = n-e, \beta_{13} = e-f+1, \gamma_{13} = f+2,$

In combined form, the pdf of 1^{st} order statistic for a *X* random variable can also be written as:

$$I(x_{1:n};\ \Psi) = nx_1\left[\sum_{g=1}^{3}\beta_g\sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f}\{\psi_{1g}x_1\}w_1^{\alpha_{1g}-1}w_2^{\beta_{1g}-1}(1-w_1-w_2)^{\gamma_{1g}-1}\right]. \quad (4.10)$$

where

$$\beta_g = \beta_1+\beta_2+\beta_3, \psi_{1g} = \psi_{11}+\psi_{12}+\psi_{13}, \alpha_{1g} = \alpha_{11}+\alpha_{12}+\alpha_{13},$$

$$\beta_{1g} = \beta_{11}+\beta_{12}+\beta_{13}, \gamma_{1g} = \gamma_{11}+\gamma_{12}+\gamma_{13}$$

4.1.3 Pdf of n^{th} order statistic of the 3-CMPD

By putting, $k = n$ in equation (4.6), the pdf of n^{th} order statistic becomes as:

$$I(x_{n:n};\ \Psi) = n\sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j\binom{n-1}{j}\binom{j}{l}\binom{l}{r}\{\beta_1(j-l)x_n\}\{\beta_2(l-r)x_n\}\{\beta_3(r)x_n\}w_1^{j-l}w_2^{l-r}(1-w_1-w_2)^r\left\{w_1\beta_1x_n^{\beta_1-1}+w_2\beta_2x_n^{\beta_2-1}+(1-w_1-w_2)\beta_3x_n^{\beta_3-1}\right\}, \quad (4.11)$$

$$I(x_{n:n};\ \Psi) = n\sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j\binom{n-1}{j}\binom{j}{l}\binom{l}{r}\{(\beta_1(j-l)+\beta_2(l-r)+\beta_3(r))x_n\}w_1^{j-l}w_2^{l-r}(1-w_1-w_2)^r\left\{w_1\beta_1x_n^{\beta_1-1}+w_2\beta_2x_n^{\beta_2-1}+(1-w_1-w_2)\beta_3x_n^{\beta_3-1}\right\},$$

$$I(x_{n:n};\ \Psi) = n\left[\beta_1\sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j\binom{n-1}{j}\binom{j}{l}\binom{l}{r}\{(\beta_1(j-l+1)+\beta_2(l-r)+\beta_3(r))x_n\}w_1^{j-l+2-1}w_2^{l-r+1-1}(1-w_1-w_2)^{r+1-1} + \beta_2\sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j\binom{n-1}{j}\binom{j}{l}\binom{l}{r}\{(\beta_1(j-l)+\beta_2(l-r+1)+\beta_3(r))x_n\}w_1^{j-l+1-1}w_2^{l-r+2-1}(1-w_1-w_2)^{r+1-1} + \beta_3\sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j\binom{n-1}{j}\binom{j}{l}\binom{l}{r}\{(\beta_1(j-l)+\beta_2(l-r)+\beta_3(r+1))x_n\}w_1^{j-l+1-1}w_2^{l-r+1-1}(1-w_1-w_2)^{r+2-1}\right], \quad (4.12)$$

where

$\psi_{21} = \beta_1(j-l+1) + \beta_2(l-r) + \beta_3(r),\ \alpha_{21} = j-l+2, \beta_{21} = l-r+1,$

$\gamma_{21} = r+1.$

$\psi_{22} = \beta_1(j-l) + \beta_2(l-r+1) + \beta_3(r),\ \alpha_{22} = j-l+1, \beta_{22} = l-r+2, \quad \gamma_{22} = r+1.$

$\psi_{23} = \beta_1(j-l) + \beta_2(l-r) + \beta_3(r+1),\ \alpha_{23} = j-l+1, \beta_{23} = l-r+1, \quad \gamma_{23} = r+2.$

In combined form, the pdf of n^{th} order statistic for a *X* random variable can also be written as:

$$I(x_{1:n};\Psi) = nx_n\left[\sum_{g=1}^{3}\beta_g\sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-)^j\binom{n-1}{j}\binom{j}{l}\binom{l}{r}\{\psi_{2g}x_n\}\right].\ w_1^{\alpha_{2g}-1}w_2^{\beta_{2g}-1}(1-w_1-w_2)^{\gamma_{2g}-1} \quad (4.13)$$

where

$$\beta_g = \beta_1+\beta_2+\beta_3, \psi_{2g} = \psi_{21}+\psi_{22}+\psi_{23}, \alpha_{2g} = \alpha_{21}+\alpha_{22}+\alpha_{23},$$

$$\beta_{2g} = \beta_{21}+\beta_{22}+\beta_{23}, \quad \gamma_{2g} = \gamma_{21}+\gamma_{22}+\gamma_{23}$$

4.1.4 r^{th} moments about origin of 1^{st} order statistics of the 3-CMPD

The r^{th} moments of 1^{st} order statistic about the origin are obtained as:

$$E(x_1^r) = \int_0^1 x_1^r I(x_{1:n};\ \Psi)\,dx, \quad (4.14)$$

By using the equation (4.9) in (4.14)

$$E(x_1^r) = n\Big[\beta_1 \sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f} w_1^{\alpha_{11}-1} w_2^{\beta_{11}-1}(1-w_1-w_2)^{\gamma_{11}-1}\int_0^1 x_1^{r+1-1}\{\psi_{11}x_1\}dx_1 + \beta_2 \sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f} w_1^{\alpha_{12}-1} w_2^{\beta_{12}-1}(1-w_1-w_2)^{\gamma_{12}-1}\int_0^1 x_1^{r+1-1}\{\psi_{12}x_1\}dx_1 + \beta_3 \sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f} w_1^{\alpha_{13}-1} w_2^{\beta_{13}-1}(1-w_1-w_2)^{\gamma_{13}-1}\int_0^1 x_1^{r+1-1}\{3x_1\}dx_1\Big], \quad (4.15)$$

In combined form, the result of r^{th} moments of 1^{st} order statistic about the origin for a *X* random variable can also be written as:

$$E(x_1^r) = n\Big[\sum_{g=1}^{3}\beta_g \sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f} w_1^{\alpha_{1g}-1} w_2^{\beta_{1g}-1}(1-w_1-w_2)^{\gamma_{1g}-1}\int_0^1 x_g^{r+1-1}\{\psi_{1g}x_g\}dx_g\Big]. \quad (4.16)$$

Where

$$\beta_g = \beta_1+\beta_2+\beta_3, \psi_{1g} = \psi_{11}+\psi_{12}+\psi_{13}, \alpha_{1g} = \alpha_{11}+\alpha_{12}+\alpha_{13},$$

$$\beta_{1g} = \beta_{11}+\beta_{12}+\beta_{13},\ \gamma_{1g} = \gamma_{11}+\gamma_{12}+\gamma_{13}$$

4.1.5 Mean and variance of 1^{st} order statistics of the 3-CMPD

By putting $r=1$ in equation (4.15), the mean of r^{th} moments of 1^{st} order statistic about the origin are obtained as:

$$E(x_1) = n\Big[\beta_1 \sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f} w_1^{\alpha_{11}-1} w_2^{\beta_{11}-1}(1-w_1-w_2)^{\gamma_{11}-1}(\psi_{11}) + \beta_2 \sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f} w_1^{\alpha_{12}-1} w_2^{\beta_{12}-1}(1-w_1-w_2)^{\gamma_{12}-1}(\psi_{12}) + \beta_3 \sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f} w_1^{\alpha_{13}-1} w_2^{\beta_{13}-1}(1-w_1-w_2)^{\gamma_{13}-1}(\psi_{13})\Big]. \quad (4.17)$$

which can also be written in combined form as:

$$E(x_1) = n\Big[\sum_{g=1}^{3}\beta_g \sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f} w_1^{\alpha_{1g}-1} w_2^{\beta_{1g}-1}(1-w_1-w_2)^{\gamma_{1g}-1}(\psi_{1g})\Big]. \quad (4.18)$$

The variance of the r^{th} moments of 1^{st} order statistic about the origin is obtained by this formula:

$$Var(x_1) = E(x_1^2) - [E(x_1)]^2, \quad (4.19)$$

After substituting the result of an equation (4.18) and when $r=2$ the result of an equation (4.15) in (4.86), the variance of r^{th} moments of 1^{st} order statistic for a *X* random variable becomes as:

$$Var(x_1) = n\left[\sum_{g=1}^{3}\sqrt{\frac{\pi}{2}}\beta_g \sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f} w_1^{\alpha_{1g}-1} w_2^{\beta_{1g}-1}(1-w_1-w_2)^{\gamma_{1g}-1}(\psi_{1g})^{\frac{3}{2}}\right] - \left[n\left\{\sum_{g=1}^{3}\beta_g \sum_{e=0}^{n-1}\sum_{f=0}^{e}\binom{n-1}{e}\binom{e}{f} w_1^{\alpha_{1g}-1} w_2^{\beta_{1g}-1}(1-w_1-w_2)^{\gamma_{1g}-1}(\psi_{1g})\right\}\right]^2. \quad (4.20)$$

4.1.6 r^{th} moments about origin of n^{th} order statistics of the 3-CMPD

The r^{th} moments of n^{th} order statistic about the origin of a 3-CMPD for a *X* r.v. are obtained as:

$$E(x_1^r) = \int_0^1 x_1^r I(x_{n:n};\ \Psi)\,dx, \quad (4.21)$$

By using the result of equation (4.85) in (4.83)

$$E(x_n^r) = n\Big[\beta_1 \sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j\binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{21}-1} w_2^{\beta_{21}-1}(1-w_1-w_2)^{\gamma_{21}-1}\int_0^1 x_1^{r+1-1}(\psi_{21})x_n dx_n + \beta_2 \sum_{j=0}^{n-1}\sum_{l=0}^{l}\sum_{r=0}^{l}(-1)^j\binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{22}-1} w_2^{\beta_{22}-1}(1-w_1-w_2)^{\gamma_{22}-1}\int_0^1 x_1^{r+1-1}\{\psi_{22}\}x_n dx_{1n} + \beta_3 \sum_{j=0}^{n-1}\sum_{l=0}^{l}\sum_{r=0}^{l}(-1)^j\binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{32}-1} w_2^{\beta_{32}-1}(1-w_1-w_2)^{3-1}\int_0^1 x_1^{r+1-1}\{\psi_{32}\}x_n dx_n\Big],$$

$$E(x_n^r) = n\Bigg[\beta_1 \sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j \binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{21}-1} w_2^{\beta_{21}-1}(1-w_1-w_2)^{\gamma_{21}-1}\, 2^{\frac{r-1}{2}}\sqrt{\frac{r-1}{2}+1}(\psi_{21})^{\frac{r-1}{2}+1} + \beta_2 \sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j \binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{22}-1} w_2^{\beta_{22}-1}(1-w_1-w_2)^{\gamma_{22}-1}\, 2^{\frac{r-1}{2}}\sqrt{\frac{r-1}{2}+1}(\psi_{22})^{\frac{r-1}{2}+1} + \beta_3 \sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j \binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{23}-1} w_2^{\beta_{23}-1}(1-w_1-w_2)^{\gamma_{31}-1} 2^{\frac{r-1}{2}}\sqrt{\frac{r-1}{2}+1}(\psi_{23})^{\frac{r-2}{2}+1}\Bigg], \quad (4.22)$$

In combined form, the result of r^{th} moments of n^{th} order statistic about the origin for a *X* random variable can also be written as:

$$E(x_n^r) = n\left[\sum_{g=1}^{3}\beta_g \sum_{e=0}^{n-1}\sum_{f=0}^{e}(-1)^j \binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{2g}-1} w_2^{\beta_{2g}-1}(1-w_1-w_2)^{\gamma_{2g}-1}\, 2^{\frac{r-1}{2}}\sqrt{\frac{r-1}{2}+1}(\psi_{2g})^{\frac{r-2}{2}+1}\right]. \quad (4.23)$$

where

$$\beta_g = \beta_1+\beta_2+\beta_3, \psi_{2g} = \psi_{21}+\psi_{22}+\psi_{23}, \alpha_{2g} = \alpha_{21}+\alpha_{22}+\alpha_{23},$$

$$\beta_{2g} = \beta_{21}+\beta_{22}+\beta_{23}, \gamma_{2g} = \gamma_{21}+\gamma_{22}+\gamma_{23}$$

4.1.7 Mean and variance of n^{th} order statistics of the 3-CMPD

By putting $r=1$ in equation (4.23), the mean of r^{th} moments of n^{th} order statistic about the origin of a 3-CMPD for a *X* r.v. are obtained as

$$E(x_n) = n\Bigg[\beta_1 \sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j \binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{21}-1} w_2^{\beta_{21}-1}(1-w_1-w_2)^{\gamma_{21}-1}(\psi_{21}) + \beta_2 \sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j \binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{22}-1} w_2^{\beta_{22}-1}(1-w_1-w_2)^{\gamma_{22}-1}(\psi_{22}) + \beta_3 \sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j \binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{23}-1} w_2^{\beta_{23}-1}(1-w_1-w_2)^{\gamma_{23}-1}(\psi_{23})\Bigg], \quad (4.24)$$

which can be written in combined form as:

$$E(x_n) = n\left[\sum_{g=1}^{3}\beta_g \sum_{e=0}^{n-1}\sum_{f=0}^{e}(-1)^j \binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{2g}-1} w_2^{\beta_{2g}-1}(1-w_1-w_2)^{\gamma_{2g}-1}(\psi_{2g})\right]. \quad (4.25)$$

The variance of the r^{th} moments of n^{th} order statistic about the origin is obtained by this formula:

$$Var(x_n) = E(x_n^2) - [E(x_n)]^2. \quad (4.26)$$

After substituting the result of an equation (4.25) and when $r=2$ the result of an equation (4.23) in (4.26), the variance of r^{th} moments of n^{th} order statistic about the origin for a *X* random variable becomes:

$$Var(x_n) = n\left[\sum_{g=1}^{3}\sqrt{\frac{\pi}{2}}\beta_g \sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j \binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{2g}-1} w_2^{\beta_{2g}-1}(1-w_1-w_2)^{\gamma_{2g}-1}(\psi_{2g})^{\frac{3}{2}}\right] - \left[n\left\{\sum_{g=1}^{3}\beta_g \sum_{j=0}^{n-1}\sum_{l=0}^{j}\sum_{r=0}^{l}(-1)^j \binom{n-1}{j}\binom{j}{l}\binom{l}{r} w_1^{\alpha_{2g}-1} w_2^{\beta_{2g}-1}(1-w_1-w_2)^{\gamma_{2g}-1}(\psi_{2g})\right\}\right]^2. \quad (4.27)$$

4.2 Estimation of Parameters of the 3-CMPD

The estimation of parameters of a 3-CMPD is discussed through the method of ML estimates.

4.2.1 Sampling scheme of the 3-CMPD

Suppose in an experiment that concerns life testing from a 3-CMPD n units are being used with a termination time *T*, which is fixed is used. An experiment was conducted, and it was shown that r units from n units failed when the time *T* was completed; however, the rest of the units that are n-*r* were still operational. We faced many real-life problems where failed items can be distinguished very clearly from which subset of the population they belong to, such as subpopulation one, subpopulation two, or three. These experiments can be found in Mendenhall and Hader (1958). For example, an engineer divided a certain number of failed objects as a part of either subpopulation one, subpopulation two, or subpopulation three based on the cause of failure. The observations made there were that from the *r* failures… $r_{1,}r_2$ and r_3 failures are belonging to subpopulation one, subpopulation two, and subpopulation three, having a reason for their failure, respectively. So, the number of uncensored observations is $r = r_1 + r_2 + r_3$ and the remaining *n-r* observations are censored and give no information about the subpopulation to which they belong. Now, define x_{ij}, $0 \leq x_{ij} \leq T$ be the failure time of the i^{th} unit belonging to the j^{th} subpopulation, where *j*= 1, 2, 3 and *i*= 1, 2, . . .,r_1.

4.2.2 The likelihood function of the 3-CMPD

The LF for a 3-CMPD for the data collection through the sampling procedure is written as:

$$L\left(\beta_1,\beta_2,\beta_3,w_{1,}w_2\big|_x\right) \propto \{\Pi_{j=1}^{r_1} w_1 f(x_{1j,})\}\{\Pi_{j=1}^{r_2} w_2 f_2(x_{2j,})\}$$
$$\{\Pi_{j=1}^{r_3}(1-w_1-w_2) f_3(x_{3j,})\}\{1-F(T)\}^{n-r}, \tag{4.28}$$

$$L\left(\beta_1,\beta_2,\beta_3,w_{1,}w_2\big|_x\right) \propto \{\Pi_{j=1}^{r_1} w_1 f(x_{1j,})\}\{\Pi_{j=1}^{r_2} w_2 f_2(x_{2j,})\}$$
$$\{\Pi_{j=1}^{r_3}(1-w_1-w_2) f_3(x_{3j,})\}\{1-F(T)\}^{n-r} \quad , \tag{4.29}$$

After simplification, the likelihood function becomes:

$$L\left(\beta_1,\beta_2,\beta_3,w_{1,}w_2\big|_x\right) \propto \sum_{K=0}^{n-r}\sum_{m=0}^{K}(-1)^K\binom{n-r}{K}\binom{K}{m}.w_1^{n-r-k+r_1}.w_2^{n-r-k+r_2}$$
$$(1-w_1-w_2)^{m+r_3}$$
$$\beta_1^{r_1} exp\left\{-\beta_1\left(\sum_{j=1}^{r_1} ln\left(\frac{1}{x_{1j}}\right)-(n-r-K)lnT\right)\right\}$$
$$\beta_2^{r_2} exp\left\{-\beta_2\left(\sum_{j=1}^{r_2} ln\left(\frac{1}{x_{2j}}\right)-(K-m)lnT\right)\right\}$$
$$\beta_3^{r_3} exp\left\{-\beta_3\left(\sum_{j=1}^{r_3} ln\left(\frac{1}{x_{3j}}\right)-mlnT\right)\right\}. \tag{4.30}$$

where

$\mathbf{x}=\left(x_{11},\ x_{12},\ldots,x_{1r_1},\ldots,x_{21},\ x_{22},\ldots,x_{2r_2},\ x_{31},\ x_{32},\ldots,x_{3r_3}\right)$ are the observed failure times, and $\Psi = (\beta_1,\beta_2,\beta_3, w_1, w_2)$.

4.2.3 ML estimators of the 3-CMPD

The unknown component and the proportion parameters $\Psi = (\beta_1,\beta_2,\beta_3, w_1, w_2)$ of the 3-CMPD are estimated by the method of ML by solving the following non-linear system of equations (4.31-4.35).

$$\frac{\partial l}{\partial\beta_1} = \frac{r_1}{\beta_1} + \sum_{j=1}^{r_1} ln(x_{1j}) - \frac{W_1(n-r)T^{\beta_1}(lnT)}{(1-W_1T^{\beta_1}-W_2T^{\beta_2}-(1-w_1-w_2)T^{\beta_3})} = 0 \tag{4.31}$$

$$\frac{\partial l}{\partial\beta_2} = \frac{r_2}{\beta_2} + \sum_{j=1}^{r_2} ln(x_{2j}) - \frac{W_2(n-r)T^{\beta_2}(lnT)}{(1-W_1T^{\beta_1}-W_2T^{\beta_2}-(1-w_1-w_2)T^{\beta_3})} = 0 \tag{4.32}$$

$$\frac{\partial l}{\partial\beta_3} = \frac{r_3}{\beta_3} + \sum_{j=1}^{r_3} ln(x_{3j}) - \frac{W_3(n-r)T^{\beta_3}(lnT)}{(1-W_1T^{\beta_1}-W_2T^{\beta_2}-(1-w_1-w_2)T^{\beta_3})} = 0 \tag{4.33}$$

$$\frac{\partial l}{\partial W_1} = \frac{r_1}{W_1} - \frac{r_3}{(1-w_1-w_2)} - \frac{(n-r)(T^{\beta_1}-T^{\beta_3})}{(1-W_1T^{\beta_1}-W_2T^{\beta_2}-(1-w_1-w_2)T^{\beta_3})} = 0 \tag{4.34}$$

$$\frac{\partial l}{\partial W_2} = \frac{r_2}{W_2} - \frac{r_3}{(1-w_1-w_2)} - \frac{(n-r)(T^{\beta_2}-T^{\beta_3})}{(1-W_1T^{\beta_1}-W_2T^{\beta_2}-(1-w_1-w_2)T^{\beta_3})} = 0 \tag{4.35}$$

It is difficult to find closed form accuracy for ML estimation. These equations cannot be solved analytically. To find the ML estimate of the factor and the ratio parameter $\beta_1,\ \beta_2,\ \beta_3,\ w_1$ and w_2. Mathematica software will be used to solve the above non-linear system of equations using some iterative numerical procedure.

Let $\Psi = (\beta_1, \beta_2, \beta_3, w_1, w_2)$ and the well-known consequence is $\hat{\Psi} \sim N\left(\Psi,\ I^{-1}(\Psi)\right)$ asymptotically. Therefore, the variance of the ML estimate depends on the main diagonal of the regression matrix. The information matrix will be derived from the expectation of the negative Hessian matrix as follows:

$$I(\beta) = -E\begin{bmatrix} \frac{\partial^2 l}{\partial \beta_1^2} & \frac{\partial^2 l}{\partial \beta_1 \partial \beta_2} & \frac{\partial^2 l}{\partial \beta_1 \partial \beta_3} & \frac{\partial^2 l}{\partial \beta_1 \partial w_1} & \frac{\partial^2 l}{\partial \beta_1 \partial w_2} \\ \frac{\partial^2 l}{\partial \beta_2 \partial \beta_1} & \frac{\partial^2 l}{\partial \beta_2^2} & \frac{\partial^2 l}{\partial \beta_2 \partial \beta_3} & \frac{\partial^2 l}{\partial \beta_2 \partial w_1} & \frac{\partial^2 l}{\partial \beta_2 \partial w_2} \\ \frac{\partial^2 l}{\partial \beta_3 \partial \beta_1} & \frac{\partial^2 l}{\partial \beta_3 \partial \beta_2} & \frac{\partial^2 l}{\partial \beta_3^2} & \frac{\partial^2 l}{\partial \beta_3 \partial w_1} & \frac{\partial^2 l}{\partial \beta_3 \partial w_2} \\ \frac{\partial^2 l}{\partial W_1 \partial \beta_1} & \frac{\partial^2 l}{\partial W_1 \partial \beta_2} & \frac{\partial^2 l}{\partial W_1 \partial \beta_3} & \frac{\partial^2 l}{\partial W_1^2} & \frac{\partial^2 l}{\partial W_1 \partial W_2} \\ \frac{\partial^2 l}{\partial W_2 \partial \beta_1} & \frac{\partial^2 l}{\partial W_2 \partial \beta_2} & \frac{\partial^2 l}{\partial W_2 \partial \beta_3} & \frac{\partial^2 l}{\partial W_2 \partial W_1} & \frac{\partial^2 l}{\partial W_2^2} \end{bmatrix}$$

where

$$\frac{\partial^2 l}{\partial \beta_1^2} = \frac{-r_1}{\beta_1^2} - \frac{W_1(n-r)(lnT)^2 T^{\beta_1}(1-W_2T^{\beta_2}-(1-W_1-W_2)T^{\beta_3})}{\left(1-W_1T^{\beta_1}-W_2T^{\beta_2}-(1-w_1-w_2)T^{\beta_3}\right)^2} \quad (4.36)$$

$$\frac{\partial^2 l}{\partial \beta_2^2} = \frac{-r_2}{\beta_2^2} - \frac{W_2(n-r)(lnT)^2 T^{\beta_2}(1-W_1T^{\beta_1}-(1-W_1-W_2)T^{\beta_3})}{\left(1-W_1T^{\beta_1}-W_2T^{\beta_2}-(1-w_1-w_2)T^{\beta_3}\right)^2} \quad (4.37)$$

$$\frac{\partial^2 l}{\partial \beta_3^2} = \frac{-r_3}{\beta_3^2} - \frac{W_3(n-r)(lnT)^2 T^{\beta_3}(1-W_1T^{\beta_1}-W_2T^{\beta_2})}{\left(1-W_1T^{\beta_1}-W_2T^{\beta_2}-(1-w_1-w_2)T^{\beta_3}\right)^2} \quad (4.38)$$

$$\frac{\partial^2 l}{\partial W_1^2} = \frac{-r_1}{W_1^2} - \frac{r_3}{(1-w_1-w_2)^2} - \frac{(n-r)(T^{\beta_1}-T^{\beta_3})^2}{\left(1-W_1T^{\beta_1}-(1-w_1-w_2)T^{\beta_3}\right)^2} \quad (4.39)$$

$$\frac{\partial^2 l}{\partial W_2^2} = \frac{-r_2}{W_2^2} - \frac{r_3}{(1-w_1-w_2)^2} - \frac{(n-r)(T^{\beta_2}-T^{\beta_3})^2}{\left(1-W_2T^{\beta_2}-(1-w_1-w_2)T^{\beta_3}\right)^2} \quad (4.40)$$

4.2.4 Limiting expressions of ML estimators of complete data set of the 3-CMPD

For obtaining the ML estimates and their variances, when the test termination time *T* approaches 1, then the observations *r,* which are uncensored, also approach the sample size n so r_1, r_2 *and* r_3 approaches to $n_1,\ n_2,\ n_3$. Thus, all observations that are monitored during the analysis process are unsupervised. As the model includes all observations as such, the details of the model increased, and the performance of the ML estimator increased.

When *T* approaches 1, the expressions of equations (4.31-4.35) become as follows:

$$\hat{\beta}_1 = \frac{n_1}{\sum ln\left(\frac{1}{x_{1j}}\right)} \quad (4.41)$$

$$\hat{\beta}_2 = \frac{n_2}{\sum ln\left(\frac{1}{x_{2j}}\right)} \quad (4.42)$$

$$\hat{\beta}_3 = \frac{n_3}{\sum ln\left(\frac{1}{x_{3j}}\right)} \quad (4.43)$$

$$\widehat{W}_1 = \frac{n_1}{n} \quad (4.44)$$

$$\widehat{W}_2 = \frac{n_2}{n} \quad (4.45)$$

4.2.5 Limiting expressions of variances of ML estimators of a complete data set of the 3-CMPD

When *T* approaches 1, the expressions of equations (4.36-4.40) become as follows:

$$Var\left(\hat{\beta}_1\big|_x\right) = \frac{n_1}{\left(\sum ln\left(\frac{1}{x_{1j}}\right)\right)^2} \quad (4.46)$$

$$Var\left(\hat{\beta}_2\big|_x\right) = \frac{n_2}{\left(\sum ln\left(\frac{1}{x_{2j}}\right)\right)^2} \quad (4.47)$$

$$Var(\hat{\beta}_3\big|_x) = \frac{n_3}{\left(\sum ln\left(\frac{1}{x_{3j}}\right)\right)^2} \quad (4.48)$$

$$Var(\hat{w}_1|_x) = \frac{n_1 n_3}{n^4} \quad (4.49)$$

$$Var(\hat{w}_2|_x) = \frac{n_2 n_3}{n^4} \quad (4.50)$$

4.2.6 Simulation Study of the 3-CMPD

As it is known that it is not possible to obtain the mathematical comparisons between ML estimators, our analysis completed this purpose using a simulation study. By taking different parameters, sample sizes, and test termination time values, the ML estimators were performed under different scenarios. Now, by using the steps of the Monte Carlo simulation study to calculate the ML estimates of the five parameters $\beta_1, \beta_2, \beta_3$, w_1 and w_2 of a 3-CMPD.

1. A sample of size n from the mixtures may be taken as:
   (i) Generate a sample of $w_1 n$ observations randomly from the first component pdf, i.e., $f_1(x;\ \beta_1)$.
   (ii) Generate a sample of $w_2 n$ observations randomly from second component pdf, i.e., $f_2(x;\ \beta_2)$.
   (iii) Generate a sample of $w_3 n$ observations randomly from third component pdf, i.e., $f_3(x;\ \beta_3)$.
2. A censored sample is selected at a test termination time *T,* which is fixed. Those observations are taken as censored ones, which are greater than fixed test termination time *T; w*hen an uncensored (complete) sample is used, this step is neglected.
3. 1000 samples are generated by using steps 1 and 2 in which sample size, the values of the parameters, and the test termination time T *are* taken as fixed.
4. By simultaneously solving the equations (4.93-4.97) and by depending upon 1000 repetitions of Monte Carlo simulation study the estimates of parameters $\beta_1, \beta_2, \beta_3$, w_1 and w_2 are calculated.

The above mentioned steps 1 to 4 were used for different values of sample size as n = 50, 100, 200, 500 for obtaining the estimates of the parameters by taking different values ($\beta_1, \beta_2, \beta_3$, w_1 , w_2)={ (2, 3, 4, 0.3, 0.5), (5, 4, 3, 0.5, 0.3)} and the values of test termination time *T* = 0.4, 0.9. In such conditions, the test termination time choice is taken, and in the obtained results of a sample, the rate of censoring is approximately equal to 10% - 25%.

Table 4.7: ML estimates of 3-CMPD with parameters $\beta_1 = 2,\ \beta_2 = 3,\ \beta_3 = 4,\ w_1 = 0.3,\ w_2 = 0.5$ and $T = 0.4,\ 0.9$ using censored data

T	n	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{w}_1$	$\hat{w}_2$
20	50	7.80089	10.61541	11.01973	0.309001	0.445384
20	100	8.14090	10.47142	11.79121	0.299475	0.446040
20	200	8.32481	10.41900	11.71900	0.295542	0.465284
20	500	8.64047	10.30591	11.49235	0.295908	0.475244
30	50	8.30870	10.50639	11.28540	0.297408	0.466077
30	100	8.572412	10.32247	11.64205	0.295157	0.475970
30	200	8.70826	10.24910	11.39890	0.296980	0.478530
30	500	8.81997	10.11887	11.19401	0.297209	0.486001

Table 4.8: ML estimates of 3-CMPD with parameters $\beta_1 = 5,\ \beta_2 = 4,\ \beta_3 = 3,\ w_1 = 0.5,\ w_2 = 0.3$ and $T = 0.4,\ 0.9$ using censored data

T	n	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{w}_1$	$\hat{w}_2$
20	50	10.95881	11.60520	10.95971	0.432387	0.305019
20	100	11.20851	11.47148	10.84121	0.449401	0.306701
20	200	11.32170	11.40986	10.55412	0.465451	0.305951
20	500	11.71638	11.29501	10.42015	0.475201	0.296541
30	50	11.41810	11.55680	10.21510	0.456050	0.299451
30	100	11.54207	11.39238	10.74209	0.478094	0.296105
30	200	11.70844	11.25910	10.43770	0.475451	0.295401
30	500	11.85159	11.13786	10.21241	0.486001	0.295199

Table 4.9: ML estimates of 3-CMPD with parameters $\beta_1 = 2,\ \beta_2 = 3,\ \beta_3 = 4,\ w_1 = 0.3,\ w_2 = 0.5$ using complete data

n	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{w}_1$	$\hat{w}_2$
50	8.88630	9.89810	10.68280	0.29900	0.49900
100	8.90350	9.93310	10.90500	0.29800	0.50000
200	8.96220	9.97070	10.94290	0.29900	0.49900
500	8.96660	9.99730	10.99860	0.29900	0.49900

Table 4.10: ML estimates of 3-CMPD with parameters $\beta_1 = 5,\ \beta_2 = 4,\ \beta_3 = 3,\ w_1 = 0.5,\ w_2 = 0.3$ using complete data

n	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{w}_1$	$\hat{w}_2$
50	11.91150	10.93810	9.87220	9.49820	0.29600
100	11.94880	10.95030	9.92910	0.49900	0.29900
200	11.95560	10.97220	9.98150	0.49900	0.29900
500	11.98630	10.98730	9.99820	0.49900	0.29900

From Tables 4.7 & 4.8, it is clear that modifications of the component and the proportion parameters with expected parameters through the method of ML estimates decrease when at a fixed test termination time *T*, the sample size increases, and the similar case exists with a huge (large) test termination time *T* as associated to the minor (small) test termination time for a sample size which is fixed. Similarly, if $\beta_1 > \beta_2 > \beta_3$ and $w_1 > w_2$, the first, the second, and the third component parameters are under-estimated (over-estimated) but the first and the second proportion parameters is over-estimated (under-estimated) at the various sizes of sample (the test termination times) for a fixed test termination time (the sample size). Furthermore, the first and the second proportion parameter is under-estimated (over-estimated); however, the second and the third (first) parameters of the component are over-estimated (under-estimated) at different test termination times (the sample sizes) for a fixed size of a sample (the test termination time *T*) when $\beta_1 < \beta_2 < \beta_3$ and $w_1 < w_2$. Under or over section size and distribution ratio are reduced for sample sizes as associated with a reduced sample size for a fixed test completion time T, and the number of overestimated or underestimated sections and ratio the coefficient with lower test completion time is greater so that T by increasing test completion time.

Tables 4.9 & 4.10 also examined the differences in the ML estimates between the component and the proportion parameters from an expected parameter decrease with a rise in the size of a sample when the test termination time *T* approaches 1. Similarly, in both cases $\beta_1 < \beta_2 < \beta_3$, $w_1 < w_2$ and $\beta_1 > \beta_2 > \beta_3$, $w_1 > w_2$ at various sizes of a sample, all the component parameters are under-estimated for a fixed size of a sample. The underestimation of a component parameter is larger for a reduced size of a sample, which is associated with the greater size of a sample. When we have less data from the sample, the weight and index of the censored data will vary more greatly in terms of under- or overestimation relative to the full data of the sample.

4.2.7 Application of Real Data Set of the 3-CMPD

Davis took a real mixture of data $s = \left(s_{11},\ s_{12},\ldots,s_{1r_1},\ s_{21},\ s_{22},\ldots,s_{2r_2},\ s_{31},\ s_{32},\ldots,s_{3r_3}\right)$ on the lifetimes of several different components that are used in an aircraft radar set. For the illustration of the proposed methodology, we take data consisting of three components: a transmitter tube of V805, a transmitter tube, and an indicator tube of V600. It is not known which component of an aircraft radar set fails until the condition of disappointment of a set of radar arises before or at a test termination time *T*= 600 hours. The tests of radar sets are directed 1340 times. The summary of the data for a test termination time *T*= 600 is given below:

$$n = 1340,\ r_1 = 866,\ r_2 = 337,\ r_3 = 83,\ r = r_1 + r_2 + r_3 = 1286,\ n - r = 54,$$

$$\sum_{j=1}^{r_1} x_{1j}^2 = 2\sum_{j=1}^{r_1} s_{1j} = 268160,\ \sum_{j=1}^{r_2} x_{2j}^2 = 2\sum_{j=1}^{r_2} s_{2j} = 100750,\ \sum_{j=1}^{r_3} x_{3j}^2 = 2\sum_{j=1}^{r_3} s_{3j} = 32500.$$

Since $n - r = 54$ we have almost 4 percent type-I censored data. However, when a test termination time *T* approaches infinity (*T* approaches 1), then the summary of a complete data set is as follows:

$$n = 1340,\ n_1 = 903,\ n_2 = 337,\ n_3 = 100,\ n = n_1 + n_2 + n_3 = 1340,$$

$$\sum_{j=1}^{n_1} x_{1j}^2 = 2\sum_{j=1}^{n_1} z_{1j} = 322660,\ \sum_{j=1}^{n_2} x_{2j}^2 = 2\sum_{j=1}^{n_2} z_{2j} = 100750,\ \sum_{j=1}^{n_3} x_{3j}^2 = 2\sum_{j=1}^{n_3} z_{3j} = 59500.$$

The ML estimates and their variances are shown in Table 4.11, given below.

Table 4.11: ML estimates and their variances using real life censored and complete data

	T	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{w}_1$	$\hat{w}_2$
ML estimates	0.4	11.87401	10.73240	15.62847	0.639813	0.239017
Variances	0.4	0.054882	0.116176	0.809115	0.000055	0.000048
ML estimates	1	12.36638	11.22623	16.24819	0.669880	0.249492
Variances	1	0.049463	0.110891	0.743750	0.000050	0.000043

From Table 4.11, we noticed that the performance of ML estimation based on unsupervised data (complete information) is significantly better than that of ML estimation based on observed data, and in addition, the results are more consistent across datasets under all the.

5. Conclusion

In order statistics, the expressions of the probability density function of k^{th} order statistics, the probability density function of 1^{st} order statistics, and the probability density function of n^{th} order statistics are computed. Also, the mathematical expressions of mean, variance, and r^{th} moments about the origin are also computed for 1^{st} and the n^{th} order statistics.

To estimate the unknown parameters of a 3-CMPD the method of ML estimation is considered. A detailed simulation study is conducted to judge the performance of the ML estimators by adopting the Monte Carlo method. The results of this simulation study are compared with the results of the complete data set. An application of real data from a 3-CMPD is taken to show the feasibility of the analysis, and based on the results, comparisons are made between the results of the complete data set and the results that are taken from the simulation study. After conducting the simulation study, it was observed that as the sample size increases, the differences between the ML estimates for component and proportion parameters and their expected values decrease when the test termination time is fixed. A similar trend is observed for larger test termination times compared to smaller ones when the sample size is held constant. Additionally, the discrepancies between the ML estimates and expected parameters diminish as the sample size grows, particularly when the test termination time (T) approaches 1. In this research, data from a real field of study is analyzed to estimate the parameters using the ML estimates method. The study's results indicated that maximum likelihood estimators based on complete (uncensored) data perform significantly better than those based on censored data. Additionally, the estimators derived from uncensored data are more dependable and precise.

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